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by

Micha Sharir

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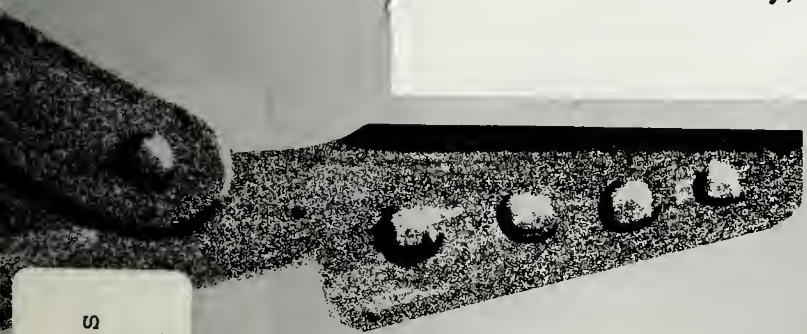

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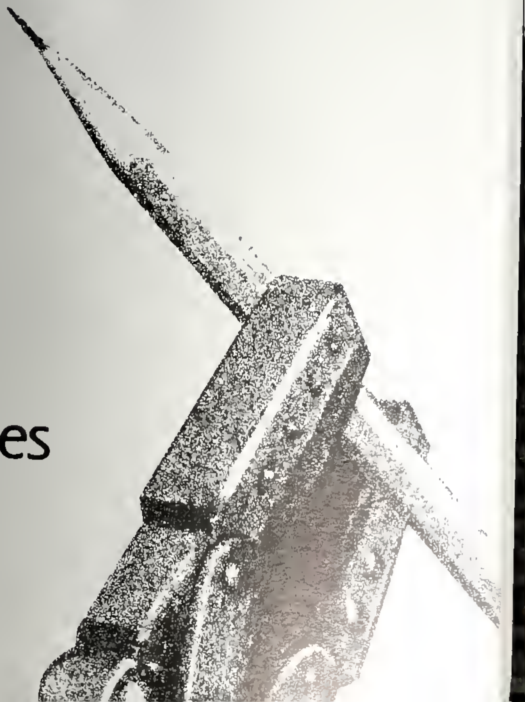

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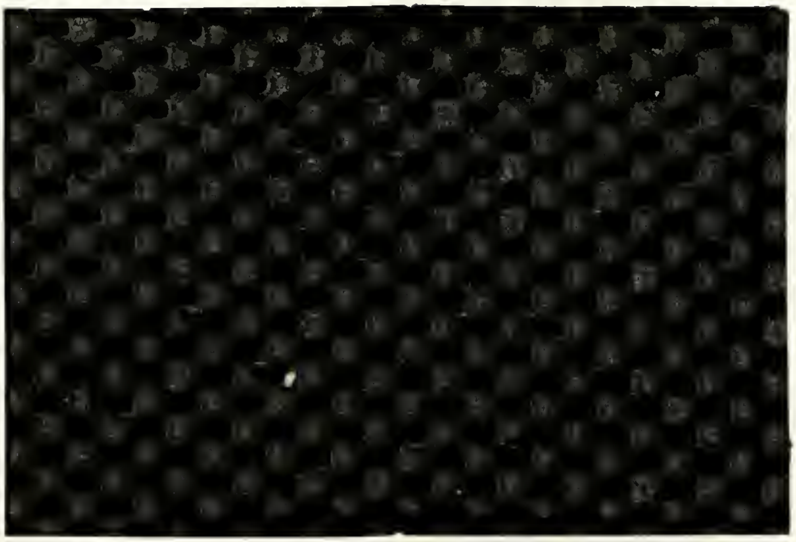

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On k-Sets in Arrangements of Curves and Surfaces

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Abstract

We extend the notion of k-sets and $(\leq k)$-sets (see [2,5,10]), to arrangements of curves and surfaces. In the case of curves in the plane, we assume that each curve is simple and separates the plane. The k-set is the set of all intersection points of the curves which are covered by exactly k interiors of (or halfplanes bounded by) other curves; the $(\leq k)$-set is the union of all j-sets, for $j \leq k$. Adapting the probabilistic analysis technique of Clarkson [5], we obtain bounds that relate the maximum size of the $(\leq k)$-set to the maximum size of a 0-set of a sample of the curves. Using known bounds on the size of such 0-sets, we obtain asymptotically tight bounds for the maximum size of the $(\leq k)$-set in the following special cases: (i) If each pair of curves intersect at most twice, the maximum size is $\Theta(nk)$. (ii) If the curves are unbounded arcs and each pair of them intersect at most three times, then the maximum size is $\Theta(nk\alpha(n/k))$. (iii) If the curves are x-monotone arcs and each pair of them intersect in at most some fixed number s of points, then the maximum size of the $(\leq k)$-set is $\Theta(k^2\lambda_s(n/k))$, where $\lambda_s(m)$ is the maximum length of (m,s)-Davenport-Schinzel sequences. We also obtain generalizations of these results to certain

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classes of surfaces in three and higher dimensions. Finally we present various applications of these results to arrangements of segments and curves, high-order Voronoi diagrams, partial stabbing of disjoint convex sets in the plane, and more.

1 Introduction

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of n closed Jordan curves, or unbounded Jordan arcs, in the plane. Let K_i denote any one of the two open regions into which γ_i separates the plane; for convenience of notation, we will call K_i the *interior* of γ_i. For a point $p \in \Re^2$, define the *level* of p, denoted $\lambda(p)$, to be the number of regions K_i containing p. Let S denote the set of all intersection points of the curves γ_i. For simplicity of exposition, we assume that these curves are in general position, meaning that no three of them meet at the same point and that no pair of them are tangent (however, our analysis can be easily modified to apply in degenerate configurations as well). For an integer $0 \le j \le n-2$, define the *j-set* of S to be

$$S_j = \{p \in S \mid \lambda(p) = j\};$$

that is, S_j is the collection of all intersection points of the curves in Γ, which are covered by exactly k interiors of other curves. Similarly, we define the *$(\le k)$-set* of S to be

$$S_{\le k} = \{p \in S \mid \lambda(p) \le k\} = \bigcup_{j \le k} S_j \ .$$

The goal of this paper is to obtain sharp bounds on the maximum size of the $(\le k)$-set. If we assume that each pair of curves in Γ intersect in at most some fixed constant number s of points, then, trivially, $|S_{\le k}| = O(n^2)$. As a matter of fact, an easy grid-like construction (see Figure 1) shows that in the worst case $|S_0| = \Theta(n^2)$. Nevertheless, in several special cases we can obtain significantly better bounds. For example, suppose that Γ is a collection of non-vertical lines, and that K_i is the upper halfplane bounded by γ_i. Then it is easily checked that S_j is the *j-level* in the arrangement $\mathcal{A}(\Gamma)$ (see Edelsbrunner [6] for more details). In this case, it is well known that $|S_{\le k}| = O(nk)$ (Alon and Györi [2]), and that this bound is tight in the worst case.

This paper explores other special cases in which bounds like $\Theta(kn)$, or slightly larger, can be obtained. We have identified three such cases. The first one, which we call the *2-intersection case*, is when each pair of curves in Γ intersect in at most two points. We show

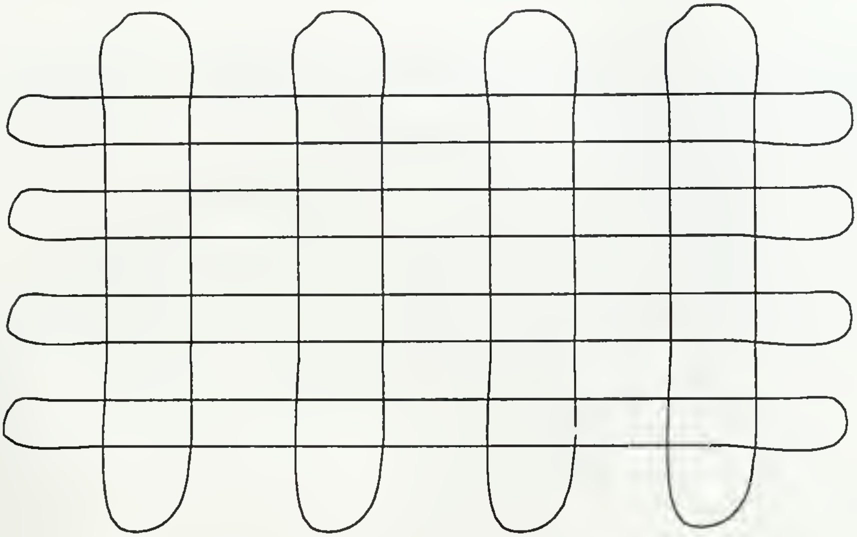

Figure 1: A grid-like arrangement of curves with $S_0 = \Theta(n^2)$

Theorem 1.1 *In the 2-intersection case, $|S_{\leq k}| \leq 26nk$, for $k \geq 1$.*

In contrast, note that in the example shown in Figure 1 the maximum number of intersections between any pair of curves is 4, and many pairs do intersect in 4 points. This still leaves one other interesting case, which we call the *3-intersection case*, where the curves in Γ are unbounded (and separate the plane); and each pair of them intersect in at most 3 points. Denoting by $\alpha(n)$ the extremely slow growing functional inverse of Ackermann's function, we show

Theorem 1.2 *In the 3-intersection case, $|S_{\leq k}| = O(nk\alpha(n/k))$, for $k \geq 1$, and this is tight in the worst case.*

Finally, we consider the *x-monotone case* where the curves in Γ are all unbounded and x-monotone, with the property that each pair of them intersect in at most s points (where s, as above, is constant). This is a natural generalization of the case of lines. Let $\lambda_s(m)$ denote the maximum length of (m, s)-Davenport-Schinzel sequences (See Hart and Sharir [11] and Agarwal et al. [1] for details). As shown in [1,11], $\lambda_s(m)$ is an almost-linear, slightly super-linear function of m for any fixed s. We show

Theorem 1.3 *In the x-monotone case, $|S_{\leq k}| = O(k^2\lambda_s(n/k))$, for $k \geq 1$, and this is tight in the worst case.*

Note that the bound in the last theorem is also very close asymptotically to kn.

The proofs of all three theorems are essentially identical. They adapt the recent probabilistic analysis technique of Clarkson [5], used originally to derive bounds on the size of ($\leq k$)-sets for point sets in any dimension (or, dually, for arrangements of hyperplanes). Clarkson's proof expresses the size of the ($\leq k$)-set in terms of the size of the 0-set in a sample of the given curves. (Note that S_0 is the set of intersection points of the curves γ_i that lie on the boundary of the union of their interiors.) Thus the availability of sharp upper bounds on the size of 0-sets facilitates the derivation of equally sharp bounds on the size of ($\leq k$)-sets. In all three cases considered in this paper we have good bounds for the size of 0-sets. In the 2-intersection case, Kedem et al. [13] have shown that $|S_0| \leq 6n$. This linear bound will imply the bound asserted in Theorem 1.1. Similarly, the 3-intersection case has been studied in Edelsbrunner et al. [7], where it was shown that $|S_0| = O(n\alpha(n))$. Again, this leads to the bound asserted in Theorem 1.2. Finally, the proof in the x-monotone case follows from the fact that S_0 is the set of all intersection points that lie on the *lower envelope* of the given curves, assuming that each K_i is the upper halfplane bounded by γ_i. The number of such "breakpoints" is known to be at most $\lambda_s(n)$ [3,11], which again leads to the bound asserted in Theorem 1.3. (The same bound holds even when some K_i's are upper halfplanes and some are lower halfplanes – see below.)

As a matter of fact, the proof technique can be generalized to higher dimensions, as long as we have sharp bounds on the complexity of 0-sets. There are a few cases where such bounds have been obtained in three and higher dimensions, and we will comment on them in the following section.

The paper is organized as follows. In Section 2 we present the proofs of Theorems 1.1–1.3 and discuss some further generalizations of the problem. In Section 3 we will present a variety of applications of our bounds to higher order Voronoi diagrams, partial stabbing of disjoint convex sets in the plane, "sparse" coverings of the plane, placements of a convex object amidst convex obstacles, and more. We conclude in Section 4 with a discussion of our results and a few open problems.

2 Upper Bounds on $(\leq k)$-Sets

Proof of Theorem 1.1: We adapt Clarkson's probabilistic technique for deriving bounds on $(\leq k)$-sets for point sets [5]. Specifically, for any subset R of Γ, let us denote by $F_j(R)$, for $j \geq 0$, the set of all intersection points of pairs of curves in R which are contained in exactly j interiors of other curves in R. (In particular, note that $F_j(\Gamma) = S_j$.)

Without loss of generality, we can assume that $k \leq n/26$. Otherwise, the asserted bound is immediate, because in this case $|S| \leq n(n-1) \leq 26k(n-1) \leq 26kn$. Let $r = \lfloor n/k \rfloor$; thus r can be assumed to be at least 26.

For each $p \in S$, let I_p denote the random variable, over the choice of a random subset $R \subset \Gamma$ of size r, whose value is 1 if $p \in F_0(R)$ and 0 otherwise. Clearly,

$$\begin{aligned}\mathbf{E}[|F_0(R)|] &= \sum_{p\in S} \mathbf{E}[I_p] = \sum_{j\geq 0}\sum_{p\in S_j} \mathbf{E}[I_p] \\ &= \sum_{j\geq 0}\sum_{p\in S_j} \mathbf{Prob}\,[p \in F_0(R)].\end{aligned}$$

The probability that a point $p \in S_j$ is in $F_0(R)$ is obtained as follows (see also [5]): Suppose $p \in \gamma_a \cap \gamma_b$. Then

(i) both γ_a and γ_b should be included in R, and

(ii) none of the j curves γ_c, for which $p \in K_c$, should be chosen in R.

The number of subsets R of Γ of size r that satisfy these conditions is $\binom{n-j-2}{r-2}$, so the probability that p is in $F_0(R)$ is $\binom{n-j-2}{r-2}/\binom{n}{r}$. Thus

$$\mathbf{E}[|F_0(R)|] \geq \sum_{j=0}^{k} |S_j| \cdot \frac{\binom{n-j-2}{r-2}}{\binom{n}{r}}.$$

But for $j \leq k$ we have

$$\frac{\binom{n-j-2}{r-2}}{\binom{n}{r}} = \frac{(n-j-2)!\; r!\; (n-r)!}{(r-2)!\; (n-j-r)!\; n!}$$

$$
\begin{aligned}
&= \frac{r(r-1)}{n(n-1)} \cdot \frac{n-r}{n-2} \cdot \frac{n-r-1}{n-3} \cdots \frac{n-r-j+1}{n-j-1} \\
&\geq \frac{r(r-1)}{n(n-1)} \cdot \left(\frac{n-r-k+1}{n-k-1}\right)^k .
\end{aligned}
$$

Since $r = \lfloor n/k \rfloor$, we easily verify that

$$\frac{n-r-k+1}{n-k-1} \geq 1 - \frac{1}{k}$$

for $k \geq 1$. Thus, for $k \geq 2$,

$$\left(\frac{n-r-k+1}{n-k-1}\right)^k \geq \left(1 - \frac{1}{k}\right)^k \geq \frac{1}{4}.$$

On the other hand, by the result of Kedem et al. [13], we have $|F_0(R)| \leq 6r$, for any subset $R \subset \Gamma$ of size r. Thus, putting everything together, we obtain

$$6r \geq \frac{r(r-1)}{4n(n-1)} \cdot \sum_{j=0}^{k} |S_j|$$

or

$$
\begin{aligned}
|S_{\leq k}| &\leq \frac{24n(n-1)}{r-1} \leq \frac{24n(n-1)}{\frac{n}{k}-2} \\
&= \frac{24nk(n-1)}{n-2k} \leq 26nk,
\end{aligned}
$$

since we assume $k \leq n/26$.

In the analysis so far we have assumed $k \geq 2$. For $k = 1$, we choose $r = \lceil n/2 \rceil$, and verify directly that

$$|S_{\leq 1}| \leq \frac{6n(n-1)(n-2)}{(r-1)(n-r)} \leq 26n$$

for all n. □

Remark: For large k, the constant of proportionality actually approaches $6e + 2 < 18.31$.

Proof of Theorem 1.2: Here we will ignore the constants of proportionality, and note that the preceding analysis can be carried out almost verbatim in this case as well, except that here we use the fact, proven in Edelsbrunner et al. [7], that

$$|F_0(R)| = O(r\alpha(r)).$$

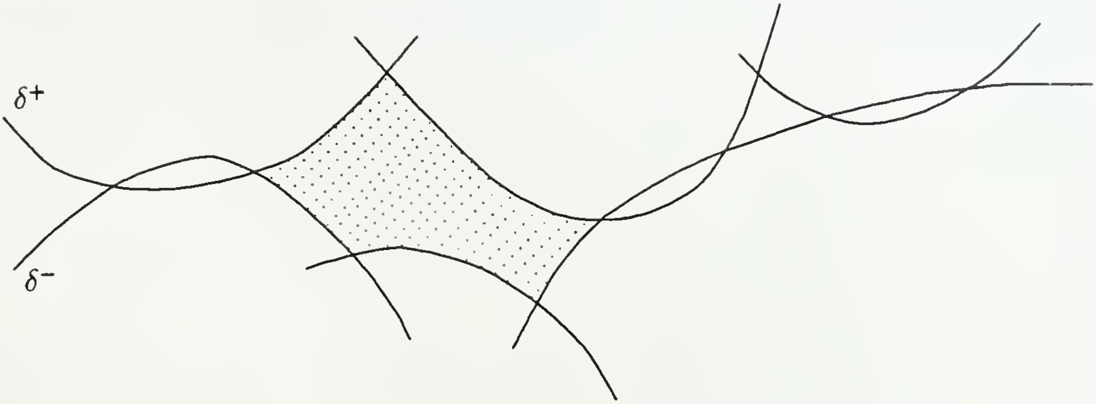

Figure 2: The region above δ^- and below δ^+, whose boundary contains the points of $F_0(R)$; one component of this region is shaded

Choosing $r = \lfloor n/k \rfloor$ as above, we readily obtain

$$|S_{\leq k}| = O(nk\alpha(n/k)).$$

The proof that this is tight in the worst case is given below. □

Proof of Theorem 1.3: Again we apply the same analysis as in Theorem 1.1. Suppose first that each K_i is the upper halfplane bounded by γ_i, $i = 1, \ldots, n$. Then $F_0(R)$ is the set of all intersection points of the curves in R which lie on the lower envelope of these curves. As is well known [3,11], the maximum size of $F_0(R)$ is $\lambda_s(r)$. Choosing $r = \lfloor n/k \rfloor$, the assertion follows. If not all K_i's are upper halfplanes, we can still show that $|F_0(R)| = O(\lambda_s(r))$. Indeed, take the lower envelope δ^+ of all curves $\gamma_j \in R$ for which K_j is an upper halfplanes, and the upper envelope δ^- of all the remaining curves. The points in $F_0(R)$ are the intersection points of curves in R which lie on the boundary of the region lying above δ^- and below δ^+ (see Figure 2), and an easy "sweeping" argument, making use of the facts that the complexity of each of these two envelopes is $O(\lambda_s(r))$, and that each pair of curves intersect in at most s points, shows that the number of such points is still $O(\lambda_s(r))$. The asserted bound on $|S_{\leq k}|$ follows. □

Remark (Lower Bounds): A lower bound of nk is known for the maximum size of $S_{\leq k}$ in an arrangement of n lines (see [2,10]). It is easy to transform this construction to obtain a similar lower bound for the 2-intersection case. For the

3-intersection case, we use the following construction. Take n/k segments whose upper envelope has combinatorial complexity $\Theta((n/k)\alpha(n/k))$; such collections of segments are constructed in Wiernik and Sharir [17]. Now replace each segment by a collection of k parallel segments lying very close to one another. Extend each segment in the new collection into an unbounded Jordan arc by two downward-directed rays emerging from its endpoints. This yields a collection of n unbounded Jordan arcs $\gamma_1, \ldots, \gamma_n$, each pair of which intersect in at most three points. Let K_i denote the open semi-infinite trapezoidal strip lying below the i-th segment (K_i is bounded by γ_i). Each intersection point of a pair of original segments which lies on their upper envelope is mapped into a $k \times k$ grid of intersection points lying very close to the original intersection. This is easily seen to imply that, with an appropriate choice of parameters, the total complexity of the ($\leq k$)-set in the final arrangement of the γ_i's is $\Omega(nk\alpha(n/k))$. A similar construction shows that the upper bound of Theorem 1.3 can also be attained. Thus in all three cases the bounds are asymptotically tight in the worst case.

Remark: The above analysis assumes that the given curves are in general position. However, we can easily modify it to apply to degenerate arrangements as well. Specifically, suppose Γ has the 2-intersection property, and consider all points p that are incident to more than two curves of Γ. Let us assume that the curves in Γ can be slightly perturbed so that they lie in general position, they still have the 2-intersection property, and, for each such p, at least one pair of them still intersect at a point sufficiently close to p. These assumptions hold in practically all applications of interest (for example, they hold in the case of circles). It follows that this perturbation does not decrease the cardinality of any j-set, so the upper bound of Theorem 1.1 continues to hold for degenerate cases as well. Similar kind of reasoning can be used to extend Theorems 1.2 and 1.3 to degenerate configurations.

We conclude this section with some observations concerning extending the results of Theorems 1.1–1.3 to ($\leq k$)-sets in arrangements of surfaces in three (or higher) dimensions. Let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a collection of surfaces in $\Re^d$ such that each surface σ_i is simple and separates $\Re^d$ into two open connected regions; let K_i denote one of these two regions. The k-set for Σ is the collection of all intersection points of d of the surfaces in Σ, which are covered by exactly k regions K_i; the ($\leq k$)-set is, as usual, the union of all j-sets, for $j = 0, \ldots, k$. We again assume general position of the surfaces σ_i.

It is easily checked (see Clarkson [5] for details) that the proof technique used in Theorems 1.1–1.3 continues to apply as long as we have a sharp bound on $|F_0(R)|$, and as long as each intersection point is determined by a fixed constant number of surfaces (which, in $\Re^d$, under the general position assumption, is d). The general bound yielded by this proof technique is easily seen to be $O(k^d|F_0(n/k)|)$. We list below two applications of this general result.

Theorem 2.1 *For a collection of n spheres in $\Re^d$ we have $|S_{\leq k}| = O(n^{d-1}k)$.*

Proof: As shown in Schwartz and Sharir [16], we have in this case $|F_0(R)| = O(r^{d-1})$. □

Theorem 2.2 *For a collection of n triangles in $\Re^3$, $\Delta_1, \ldots, \Delta_n$, let K_i denote the semi-infinite vertical triangular prism whose upper boundary is Δ_i, for $i = 1, \ldots, n$ (the corresponding surface σ_i is the union of Δ_i with the three vertical faces of K_i). In this case we have $|S_{\leq k}| = O(n^2 k\alpha(n/k))$. In particular, the number of triple intersections of the given triangles, which lie below no more than k other triangles, is also $O(n^2 k\alpha(n/k))$.*

Proof: As shown in Pach and Sharir [15], we have in this case $|F_0(R)| = O(r^2\alpha(r))$. □

3 Applications

In this section we obtain a variety of applications of the preceding results.

3.1 Sparse coverings

Theorem 3.1 *If no point of the plane is covered by more than k regions K_i, then the total complexity of the arrangement $\mathcal{A}(\Gamma)$ is $O(nk)$ in the 2-intersection case, and $O(nk\alpha(n/k))$ in the 3-intersection case.*

Proof: Obvious. □

Remark: J. Pach has provided the following direct proof of Theorem 3.1 for the special case of discs. The proof proceeds by induction on n, starting with $n = k$, and shows that a circle $\gamma \in \Gamma$ having minimum radius cuts at most $O(k)$ other circles. Removing γ from Γ and applying the induction hypothesis, the theorem follows.

To establish the above claim, let K denote the disc bounded by γ, and let $K^\star$ denote a disc concentric with K and having radius three times larger than that of K. If $\gamma \cap \gamma_i \neq \emptyset$ then the area of $K_i \cap K^\star$ is greater than or equal to the area of K. Since no point of $K^\star$ is covered by more than k of the given discs, the number of such K_i is at most $9k$.

Theorem 3.2 *Let $e_1, \ldots, e_n$ be n non-vertical segments in the plane with the property that no vertical line cuts more than k of them. Then the total number of intersections of these segments is $O(nk\alpha(n/k))$. Similarly, let $\gamma_1, \ldots, \gamma_n$ be n x-monotone bounded arcs in the plane, each pair of which intersect in at most s points, with the additional property that no vertical line cuts more than k of these arcs. Then the total number of intersections of these arcs is $O(k^2\lambda_{s+2}(n/k))$.*

Proof: In the case of segments, reduce this configuration to the 3-intersection case, as in the construction in the lower bound remark given in the preceding section, and apply Theorem 3.1. In the case of arcs, a similar extension to full unbounded curves (each pair of which, by a well known observation [3], intersect in at most $s+2$ points) and an application of Theorem 1.3 complete the proof. □

Corollary 3.3 *The combinatorial complexity of the arrangement of k x-monotone polygonal curves, consisting of a total of n segments, is $O(nk\alpha(n/k))$.*

Remark: We do not know for which values of k the bounds in Theorem 3.1 (the 3-intersection case), Theorem 3.2 and Corollary 3.3 are tight in the worst case (they are not tight for $k = 1$, for example).

Corollary 3.4 *Let $\Delta_1, \ldots, \Delta_n$ be n non-vertical triangles in 3-space with the property that no vertical line cuts more than k of them. Then the total combinatorial complexity of their arrangement is $O(n^2k\alpha(n/k))$. In particular, the combinatorial complexity of the arrangement of k piecewise-linear* terrains *(a terrain is a surface meeting each vertical line in exactly one point) in 3-space, having a total of n faces, is $O(n^2k\alpha(n/k))$.*

The two preceding corollaries can be applied to obtain the following type of result. Let σ be a piecewise-linear terrain in 3-space, having n faces. Let B be a *k-legged robot*; for simplicity, assume B is a rigid collection of k vertical line segments, whose top endpoints all lie on a common horizontal plate (however, the results to be stated below also apply to more general robots). We want to find all translations of B at which three of its legs touch σ (at their bottom endpoints), while other legs might "pierce" through σ, or remain above it. Assuming general position of B and σ, the number of such placements is finite. To obtain such placements, we use the standard technique (see [8] for details) of forming the Minkowski differences $\sigma - l_i$, where l_i is the bottom endpoint of the i-th leg of B, $i = 1, \ldots, k$. Each of this differences is just a translation of σ, and the placements we seek correspond to vertices of the arrangement of these k copies of σ. The preceding corollary now yields the following result. We state its two-dimensional version as well, which is obtained in a completely analogous manner, making use of Corollary 3.3.

Corollary 3.5 *The maximum number of translated placements of triple contact of a k-legged robot with a piecewise-linear terrain with n faces, as defined above, is $O(n^2 k\alpha(n/k))$. In two dimensions, the maximum number of translated placements of double contact of a k-legged robot with an x-monotone polygonal curve consisting of n segments is $O(nk\alpha(n/k))$.*

3.2 Placements of convex objects

Theorem 3.6 *Let $A_1, \ldots, A_n$ be n convex sets in the plane having pairwise disjoint interiors, and let B be another convex set. Assuming general position, the maximum number of translated positions of B, at which it simultaneously touches two of the sets A_i, and otherwise intersecting at most k other such sets, is $O(nk)$.*

Proof: This is an immediate consequence of the analysis in [13], combined with Theorem 1.1. Specifically, we form the Minkowski differences $K_i = A_i - B$, $i = 1, \ldots, n$, and observe, as in [13], that each of the desired placements of B corresponds to an intersection point of the boundaries of two sets K_i, which is covered by at most k other such sets. Since, as shown in [13], the boundaries of any pair of sets K_i intersect at most twice (assuming general position), the claim follows immediately from Theorem 1.1. □

3.3 Partial stabbing

Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a collection of n pairwise disjoint compact convex sets in the plane. For each line ℓ, let $\sigma(\ell)$ denote the sequence of those sets C_i that intersect ℓ, ordered in their order along ℓ (we will identify $\sigma(\ell)$ with its reverse sequence). A line ℓ is a *j-stabber* of $\mathcal{C}$ if $|\sigma(\ell)| = j$; an n-stabber is also called a *common transversal* of $\mathcal{C}$. Edelsbrunner and Sharir [9] have shown that the maximum number of distinct sequences $\sigma(\ell)$, over all n-stabbers ℓ of $\mathcal{C}$, is $2n-2$. We can extend this result to show

Theorem 3.7 *The maximum number of distinct sequences $\sigma(\ell)$, over all j-stabbers ℓ of $\mathcal{C}$, for $j = n, n-1, \ldots, n-k+1$, is $O(nk)$. This is tight in the worst case.*

Proof: Let ℓ be a j-stabber of $\mathcal{C}$. Arguing as in Lemma 1 of [9], one can show that ℓ can be continuously moved to an *extreme line* ℓ^*, such that $\sigma(\ell) = \sigma(\ell^*)$ and ℓ^* is tangent to two sets $C, C' \in \mathcal{C}$ that lie on the same side of ℓ^*. Thus it suffices to obtain the asserted bound for the total number of extreme j-stabbers, over all $j = n, n-1, \ldots, n-k+1$.

To this end, we use duality, as in [9]. In the dual plane, each C_i is mapped into a pair of unbounded x-monotone curves γ_i^-, γ_i^+, such that γ_i^- is concave, γ_i^+ is convex, and γ_i^- lies below γ_i^+. Moreover, a point p lies in the region between γ_i^- and γ_i^+ if and only if its dual line p^* stabs C_i. Points $p \in \gamma_i^-$ (resp. γ_i^+) correspond to lines p^* that are tangent to C_i and lie below (resp. above) it. Moreover, as observed in [9], for any $i \neq j$ we have

$$|\gamma_i^- \cap \gamma_j^-|\,,\ |\gamma_i^- \cap \gamma_j^+|\,,\ |\gamma_i^+ \cap \gamma_j^+| \leq 2$$

(think of the corresponding property in the primal plane). For each i, let K_i^- denote the halfplane below γ_i^-, and let K_i^+ denote the halfplane above γ_i^+. Note that an extreme j-stabber p^* is the dual of an intersection point p between two upper curves γ_a^+, γ_b^+, or between two lower curves γ_a^-, γ_b^-, which lies between γ_q^- and γ_q^+ for at least $n-k-1$ other indices q. Thus p is covered by at most $2k$ regions K_i^-, K_i^+, $i = 1, \ldots, n$. This, the 2-intersection property of the curves γ_i^-, γ_i^+, and Theorem 1.1 imply that the total number of desired extreme stabbers is $O(kn)$. The asserted upper bound in the theorem is now immediate. Moreoevr, in the construction given in [9] (originally presented in [12]), the number of j-stabbers, $j = n, n-1, \ldots, n-k+1$ is easily seen to be $\Omega(nk)$, showing that the asserted bound in tight in the worst case. □

3.4 High-order Voronoi diagrams

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in the plane. The j-th order Voronoi diagram $Vor_j(P)$ of P is a convex subdivision of the plane, each of whose regions is associated with a subset $T \subset P$ of cardinality j, and is denoted as $V(T)$, such that

$$V(T) = \{x \in \Re^2 \mid d(x,p) < d(x,p') \text{ for all } p \in T, p' \in P - T\} .$$

See [6] for more details concerning high-order Voronoi diagrams.

It is well known that the overall complexity of the diagrams $Vor_j(P)$, $j = 1, \ldots, k$, is $O(k^2 n)$ [6]. However, we can obtain a refinement of this bound in the following sense. Let e be an edge of the j-th order Voronoi diagram, bounding the regions $V(T)$, $V(T')$. For each $x \in e$, let us define

$$d_j(x) = \max_{p \in T} d(x,p) = \max_{p \in T'} d(x,p) .$$

Then we have

Theorem 3.8 *Let $a \in \Re^+$ be a fixed positive number. The maximum number of edges $e \in Vor_j(P)$, over all $j \leq k$, which contain an interior point x with $d_j(x) = a$, is $O(kn)$.*

Proof: For each $p_i \in P$, let D_i be the disc of radius a centered at p_i. It is immediate from the definitions that a point $x \in e \in Vor_j(P)$ with the above properties is an intersection point of the bounding circles of two discs D_i, $D_{i'}$ which also lies in the interior of $j-1$ other discs. The assertion is now immediate from Theorem 1.1. □

Proof: This can be generalized to high-order diagrams defined by an arbitrary metric (or even a "convex distance function" [14]), and by an arbitrary collection P of closed, convex and pairwise disjoint objects.

3.5 Algorithmic issues

Finally, we consider the problem of computing efficiently the $(\leq k)$-set in an arrangement of n given curves of one of the types considered above. The following simple divide-and-conquer approach may be used. We describe it for the 2-intersection case only, but the technique extends trivially to the 3-intersection and the x-monotone cases. In what follows we assume a model of computation in which various basic operations on the given curves can be performed in constant time. Typical such operations are: finding the intersection points of pairs of the curves, breaking each curve into x-monotone portions, and testing whether a given point lies above, on, or below a given x-monotone curve portion.

Divide the given collection Γ into two subsets Γ_1, Γ_2 of roughly equal size. Recursively compute the $(\leq k)$-sets for Γ_1, Γ_2, respectively. More precisely, the output of the recursive processing of Γ_1 is the planar map $\mathcal{M}_1$ formed by all the faces of the arrangement $\mathcal{A}(\Gamma_1)$, which are covered by at most k interiors of curves in that set; we also assume that each face of $\mathcal{M}_1$ is labeled with the number of interiors covering it. The output of processing Γ_2 is a similar map $\mathcal{M}_2$, defined in an analogous manner.

Next we merge $\mathcal{M}_1$ and $\mathcal{M}_2$ using a standard plane-sweeping technique. It is easily checked that each face in the desired output map $\mathcal{M}$ for the whole set Γ must be obtained as a connected component of the intersection of a face of $\mathcal{M}_1$ and a face of $\mathcal{M}_2$. Moreover, every intersection between an edge of $\mathcal{M}_1$ and an edge of $\mathcal{M}_2$ must belong to the $(\leq 2k)$-set of Γ. Since this set has size $O(kn)$, it follows that the merging of $\mathcal{M}_1$ and $\mathcal{M}_2$ to produce $\mathcal{M}$ can be done in time $O(kn \log n)$. We stop the recursion when $n \leq k$, in which case we compute the entire arrangement $\mathcal{A}(\Gamma)$, in time $O(k^2 \log k)$. It follows that the overall time complexity of this algorithm is $O(nk \log n \log \frac{n}{k})$. We thus summarize

Theorem 3.9 *In the 2-intersection case, one can calculate the $(\leq k)$-set in an arrangement of n curves in time $O(nk \log n \log \frac{n}{k})$. In the 3-intersection case, this takes time $O(nk\alpha(n/k) \log n \log \frac{n}{k})$. In the x-monotone case, this takes time $O(k^2 \lambda_s(n/k) \log n \log \frac{n}{k})$.*

4 Conclusion

In this paper we have made the simple observation that Clarkson's probabilistic analysis technique can be adapted to yield sharp bounds on the size of $(\leq k)$-sets in arrangements of curves and surfaces, and have applied it to a variety of problems.

We expect that our results will have many more applications. One such application, to be presented in a forthcoming paper, yields an efficient algorithm for stabbing a given collection of discs by query points. We leave it as a challenge to find other interesting applications.

A main remaining open problem is to obtain sharp upper bounds on the size of k-sets in such arrangements. For example, can the bound $O(n\sqrt{k})$, which is known for arrangements of lines (see e.g. [6]), be established in the other cases studied in this paper?

Another interesting problem is to identify cases in which we do not have a sharp uniform bound on the size of $|F_0(R)|$, but can still apply the probabilistic proof technique by obtaining a sharp bound on the *expected* size of $F_0(R)$. This might be possible, for instance, in the case of arbitrary arrangements of curves such that the total number of intersections between them is small.

Finally, the statements of some of the results obtained here seem to be sufficiently simple to warrant a direct, non-probabilistic, proof, such as Pach's proof of the special case of Theorem 3.1. It would be interesting to obtain other simple proofs, e.g. of Theorem 3.2.

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